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The related properties and application of orthogonal matrix

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Abstract

This paper mainly introduces the definition of orthogonal matrix and its nine common properties, as well as some applications of orthogonal matrix in advanced algebra: using orthogonal matrix to solve some specific problems related to orthogonal transformation in Euclidean space, transforming quadratic form into canonical form through orthogonal matrix, simplifying quadratic surface equation with orthogonal matrix, and using orthogonal matrix in matrix decomposition.

Keywords:orthogonal matrix; quadratic form; matrix decomposition.

Reword

Orthogonal matrix is an important kind of matrix theory, which is more widely used than other modules. In particular, orthogonal matrices often play a significant role in abstract proof problems, such as positive definite matrices. In addition, for some problems in Euclidean space, after taking the orthogonal basis, the problem can be solved with the orthogonal matrix. In the thesis of postgraduate entrance examination, it is not difficult to investigate some of the questions related to the orthogonal matrix. It is common to use the orthogonal matrix to solve the problem of diagonalization. At the same time, the orthogonal matrix also has some applications in the recent algebra. Not only in the field of mathematics, but also in physics and chemistry. For example, in chemistry, the role of the orthogonal matrix in the atomic orbital hybridization, and the physical aspects of the rigid body motion. Thus, the application of the orthogonal matrix is indeed very extensive. The key to use the orthogonal matrix to solve related problems is to understand the definition and related judgment of orthogonal matrix, master the properties of orthogonal matrix, and use them reasonably, combined with some common applications related to orthogonal matrix, so as to obtain more profound insights into orthogonal matrix. Orthogonal matrix properties and many aspects of the application of many scholars have a deeper research, the study, the author on the basis of the conclusion of predecessors and summary, application aspects mainly around the related parts of higher algebra, hope this article for subsequent research scholars can play a certain help.

1 Orthogonal matrix and its associated definitions

1.1 Definition of the orthogonal matrix

Definition 1: a $A^T A = AA^T = E$ satisfied A matrix, called an orthogonal matrix.

In particular, a particular matrix can also serve as an orthogonal matrix, as follows:

Special form [1]: n The order matrix E composed n of the column n vectors of the order unit matrix is called the permutation matrix. And such a permutation matrix is clearly an orthogonal matrix.

1.2 Judgment of the orthogonal matrix

Theorem: when $A=\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$ it n is a step real square matrix, there are the following conditional equivalence [1]:

1) It A is an orthogonal matrix;
2) It A^T is an orthogonal matrix;
3) It A^* is an orthogonal matrix;
4) is A^{-1} the orthogonal matrix.
5) ; $A^T=A^{-1}$
6) It $-A$ is an orthogonal matrix;
7) Can $\alpha_1,\alpha_2,\ldots,\alpha_n$ be used as a set R^n of orthogormal bases of Euclidean space;
8) each A^T row is a unit vector and two orthogonal;

T A^T he columns are the unit vector and are pairwise orthogonal;

Take the third-order orthogonal matrix as an example:

If it $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is an orthogonal matrix, there are:

, $a_{11}^2+a_{21}^2+a_{31}^2=a_{12}^2+a_{22}^2+a_{32}^2=a_{13}^2+a_{23}^2+a_{33}^2=1$ a word $a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32}=0$ used in a person's name.

From the above several judgment conditions, we can conclude that the diagonal matrix is an ± 1 orthogonal matrix: the elements on the main diagonal are.

2 Correlation properties of the orthogonal matrix

When A it is an orthogonal matrix, there are the following properties [2][3]:

If property 1 A,B is orthogonal matrix, $|A|=\pm 1,|B|=\pm 1$ and all AB is orthogonal matrix.

Proof: by as A,B an orthogonal matrix, then, available

$$AA^T=E;BB^T=E.$$

$$,(AB)^T AB=B^T A^T AB=E$$

Thus, it AB is an orthogonal matrix.

If property 2 A is an orthogonal matrix, the modulus length of each eigenvalue is 1, and the eigenvectors belonging to different eigenvalues are orthogonal to each other.

Evidence: λ set to any eigenvalue A of the orthogonal α matrix A, λ and is the eigenvector corresponding to, namely, the following equation holds:

$$,A^T A=AA^T=E,A\alpha=\lambda\alpha$$

By the conjugate transpose method, get

$,\bar{\alpha}^T A^T A\alpha=\bar{\lambda}\bar{\alpha}^T A\alpha$ Where $\bar{\lambda},\bar{\alpha}$ the conjugate λ,α.

Therefore, $\lambda\bar{\lambda}\bar{\alpha}^T\alpha=\bar{\alpha}^T\alpha$ yes, the combined eigenvector cannot $|\lambda|=1$ be zero λ vector, so, the A arbitrariness of the combination, the module length of each eigenvalue is 1.

According A to the orthogonal λ,μ matrix A, can be set as any two x,y different λ,μ eigenvalues, respectively for the corresponding eigenvectors, so, there are

$$(Ax=\lambda x\ 1),$$

$$,Ay = \mu y$$

Take the right side at the same time, have

$$（ y^T A^T = \mu y^T 2）$$

Multiply (2) with (1), get

$,(\lambda\mu-1)(x,y)=0$ Since, $|\lambda|=|\mu|=1$ and $\lambda \neq \mu$

So, so. $(\lambda\mu-1)\neq 0$ $(x,y)=0$

In summary, the different eigenvalues of the orthogonal matrix correspond to the eigenvectors that are mutually orthogonal.

Properties 3 assumes A an orthogonal matrix as follows:

If, there $|A|=-1$ is a characteristic value of-1.

If it $|A|=1$ is odd n, there must be an eigenvalue of 1.

Proof:, by, $|A+E|=|A+E|^T$ get

$$,|A+E|^T=\left|(A+E)^T\right|=\left|A^T+E\right|=\left|A^T\right|\cdot|E+A|$$

And, $|A|=-1=\left|A^T\right|$ so:

$$.|A+E|=-|A+E|$$

approach $|A+E|=0$. Thus, -1 A is the eigenvalue $|E-A|=|E-A|^T$ of. Because, I get

$$,|E-A|^T=\left|(E-A)^T\right|=\left|E-A^T\right|=\left|A^T\right|\cdot|A-E|$$

And, $|A|=1=\left|A^T\right|$ and n is odd, so:

$$.|E-A|=-|E-A|$$

approach $|E-A|=0$. Thus, 1 A is the characteristic value of.

If property 4 A is n an orthogonal α matrix and R^n a column vector in Euclidean $\|A\alpha\|=\|\alpha\|$ space, it is.

Proof: Since, A as an orthogonal matrix, there is:

Let the corresponding A matrix of the orthogonal transformation A under a certain set of orthogonal bases be, then, according to the nature of the orthogonal transformation, there is

$$(\mathrm{A}\, a,\mathrm{A}\, a)=(a,a)$$

And, $(\mathrm{A}\, a,\mathrm{A}\, a)=(Aa,Aa)$ yes $(Aa,Aa)=(a,a)$, that $\|Aa\|=\|a\|$ is.

Property 5 should A be n an order orthogonal matrix, then for n any order B orthogonal matrix $Tr(ABA^T)=Tr(B)$.

Evidence: Since it is A an orthogonal matrix, so, there is $AA^T=E$, i. e $A^T=A^{-1}$. Therefore $ABA^T=ABA^{-1}$, that ABA^T is similar B, the bound similarity matrix has the same eigenvalue, set B the eigenvalue as $\lambda_i(i=1,2,\ldots,s)$ and, therefore $Tr(B)=Tr\left(ABA^T\right)$ $Tr(B)=\lambda_1+\lambda_2+\cdots+\lambda_s$.

Property 6 assumes A both a symmetric and an orthogonal matrix, then we call A this matrix the A alignment matrix ±1, and thus the eigenvalue can only be.

Prove: by is A a symmetric matrix, so, there is

$$;A^T=A$$

By the A orthogonal matrix, so, $AA^T=E$ there $A^T=A^{-1}$ is, i. e.

To sum up, $A^T=A^{-1}=A$ get, combine $A^TA=E$, $A^2=E$ and A so. It is therefore the $A^2=E$ alignment α matrix A. λ Again, let it be the $A^2\alpha=A(A\alpha)=\lambda^2\alpha=\alpha$ eigenvalue $\lambda^2=1$ corresponding $\lambda=\pm1$ to the eigenvector, then. I. e., or.

In summary A, the eigenvalues ±1 can only be.

Property 7 If A an orthogonal matrix of the upper (lower) triangle, it must A be a diagonal matrix ± 1, and the elements on the main diagonal element are.

Proof: The following A case is the orthogonal matrix of the lower triangle, the case is similar to the upper triangle.

By the A trigonal orthogonal matrix, so $(A \vdots E)$ the elementary row transformation $\left(E \vdots A^{-1}\right)$, thus A^{-1}, know is under the triangle orthogonal $AA^T = E$ matrix $A^T = A^{-1}$, A^T and, namely, namely both the triangle A matrix, and the triangle A matrix, diagonal matrix, and A by the column, the vector ± 1 module length is 1, so the elements on the main diagonal element is.

Property 8 [3] A If the orthogonal matrix is an asymmetric A matrix, then the eigenvalues cannot be all real numbers.

Evidence: Starting from the counterproof method A, set the eigenvalue of the orthogonal Q matrix $Q^T AQ = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$ are A all real numbers, so there is an orthogonal matrix, so that the A symmetric matrix (diagonal matrix is a special symmetric matrix), so it should be a symmetrical orthogonal matrix, which is inconsistent with the known.

Therefore, the eigenvalues of A a matrix known to be all real.

Property 9 [3] A set is an orthogonal $|A| = 1$ matrix A, if, the arbitrary child is equal $|A| = -1$ to its A algebraic coactor; if, the arbitrary child is only one negative sign from its algebraic coactor.

Evidence: A starting from the orthogonal matrix

$$, A^T = A^{-1} = \frac{1}{|A|} A^*$$

So, at the time $1 \le i, j \le n$, $A_{(i,j)} = A^T_{(i,j)} = \frac{1}{|A|} A^*_{(i,j)} = \frac{1}{|A|} A_{ij}$

From the above formula, you can get, the proposition holds.

The properties of the orthogonal matrix are far more than the above ones, but the above properties related to the orthogonal matrix are relatively common and widely used. Flexible use of the above properties can solve many problems related to the orthogonal matrix.

3 Application of the orthogonal matrix

As a special class of matrix theory, orthogonal matrix has a wide range of applications, not only in mathematics, orthogonal matrix also has certain applications in physics, chemistry and other aspects. Based on the author's own knowledge reserve, the application of orthogonal matrix is mainly expanded by the higher algebra part in mathematics, which is divided into three modules: the application of orthogonal matrix in Euclidean space, quadratic standard shape and matrix decomposition.

Before this, the author first discusses some problems of the relatively simple orthogonal matrix, and then expands the relevant application of the remaining three parts.

Example 1 proof: no order n orthogonal A matrix B, $A^2 = AB + B^2$ so.

Prove: (counterproof method) n may set the A, B existence of order orthogonal matrix, making

$$A^2 = AB + B^2 \qquad (1)$$

Deform the (1) twice:

$)i$ By the right B^{-1}, yes $A + B = A^2 B^{-1}$,

$)ii$ Deform the (1) $A\left(A - B\right) = B^2$, yes, and A^{-1} then $A - B = A^{-1}B^2$ multiply by the left,

Combination A, B is orthogonal matrix, $A^2 B^{-1}$ by property 1, $A + B$ know is orthogonal matrix $A - B$, that is, orthogonal matrix, similarly, there is orthogonal matrix, then, by the definition of orthogonal matrix, get

$$E=\left(A+B\right)^{T}\left(A+B\right)=2E+A^{T}B+B^{T}A$$

$$E=\left(A-B\right)^{T}\left(A-B\right)=2E-A^{T}B-B^{T}A$$

The addition of $2E=4E$ the two types, get, contradiction, so the conclusion is established.

Example 2 Let the block $A=\begin{pmatrix} P & R \\ O & Q \end{pmatrix}$ matrix be an P orthogonal m matrix Q, n which is P,Q an order matrix $R=O$, an order matrix, and an orthogonal matrix, and.

Prove: from the question A, is an orthogonal matrix, so

$$,A^{T}A=\begin{pmatrix} P^{T} & O \\ R^{T} & Q^{T} \end{pmatrix}\begin{pmatrix} P & R \\ O & Q \end{pmatrix}=\begin{pmatrix} P^{T}P & P^{T}R \\ R^{T}P & P^{T}R+Q^{T}Q \end{pmatrix}=E_{m+n}$$

So, get

$$P^{T}P=E_{m},P^{T}R=O,R^{T}P=O,\quad R^{T}R+Q^{T}Q=E_{n}$$

From the above $P^{T}P=E_{m}$, $R=O$ yes $Q^{T}Q=E_{n}$,, P,Q . so is the orthogonal $R=O$ matrix, and.

The above simple problems show that the definition and properties of the orthogonal matrix play a great role in solving the problems related to the same orthogonal matrix. The following follows the application of each parts from a typical problem.

3.1 Application of orthogonal matrix in Euclidean space

This part is mainly reflected in using the orthogonal matrix to solve some problems in the Euclidean space. In Euclidean space, a set of orthoronormal bases of this space is taken, and the problem of transformation is transformed into a matrix problem. In addition, there is the transition from base to base, the common is a set of orthonogonal bases to another set of orthonogonal bases, which will be reflected in practical examples.

Example η 3 n set is a unit V vector in dimensional Euclidian space, defining the formula:

$$,\mathscr{A}\left(\alpha\right)=\alpha-2\left(\eta,\alpha\right)\eta,\left(\forall\alpha\in V\right)$$

Evidence $\mathscr{A}$ is an orthogonal transformation of the second class.

Proof: The following advance $\mathscr{A}$ proof V is a linear transformation $\alpha,\beta\in V$, . $k\in R$ For any arbitrary, by the problem set the condition, know

$$\begin{aligned} &\mathscr{A}\left(\alpha+\beta\right)=\left(\alpha+\beta\right)-2\left(\eta,\alpha+\beta\right)\eta \\ &=\left(\alpha-2\left(\eta,\alpha\right)\eta\right)+\left(\beta-2\left(\eta,\beta\right)\eta\right) \\ &=\mathscr{A}\left(\alpha\right)+\mathscr{A}\left(\beta\right) \\ &\mathscr{A}\left(k\alpha\right)=k\alpha-2\left(\eta,k\alpha\right)\eta=k\left(\alpha-2\left(\eta,\alpha\right)\eta\right) \\ &=k\mathscr{A}\left(\alpha\right) \end{aligned}$$

Thus is $\mathscr{A}$ a linear transformation.

It is known to η be n the unit vector in viaulidian space, and η then V expanded to a set of $\eta,\gamma_2,\cdots,\gamma_n$ orthogonal $\mathscr{A}\left(\alpha\right)=\alpha-2\left(\eta,\alpha\right)\eta$ bases, combined, yes

$$\mathscr{A}\left(\eta\right)=\eta-2\left(\eta,\eta\right)\eta=-\eta,$$

$$\mathscr{A}\left(\gamma_i\right)=\gamma_i-\left(\eta,\gamma_i\right)\eta=\gamma_i,\left(i=2,3,\ldots,n\right),$$

thereupon $\mathscr{A}\left(\eta,\gamma_2,\gamma_3,\ldots,\gamma_n\right)=\left(\eta,\gamma_2,\gamma_3,\ldots,\gamma_n\right)A$.

Where, $A=\begin{pmatrix}-1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 1\end{pmatrix}$ is an orthogonal matrix A , and therefore is an orthogonal transformation,

Combination $|A|=-1$, A and then for the orthogonal transformation of the second type.

Example $\eta_1,\eta_2,...,\eta_n$ 4 is set as a V set of orthogonal bases in Euclidean space, and has

$$,(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)=(\eta_1,\eta_2,...,\eta_n)A$$

The A necessary condition for the orthogonal $\varepsilon_1,\varepsilon_2,...,\varepsilon_n$ matrix V is: yes a set of orthogonal bases.

Prove [4]: ε_i the V coordinate under $\eta_1,\eta_2,...,\eta_n$ the α_i orthnormal A base i is the first column, so have $(\varepsilon_i,\varepsilon_j)=\alpha_i^T\alpha_j, i,j=1,2,...,n$

Necessity: Starting from A the orthogonal matrix, from the orthogonal A matrix $=(\alpha_1,\alpha_2,...,\alpha_n)$, This is the $\alpha_1,\alpha_2,...,\alpha_n$ orthogonal unit vector $\alpha_i^T\alpha_j$, δ_{ij} with $(\varepsilon_i,\varepsilon_j)=$, $=$ and δ_{ij} then δ_{ij} (indicating $i\neq j$: $\delta_{ij}=0$ then $i=j$,; $\delta_{ij}=1$ then $\varepsilon_1,\varepsilon_2,...,\varepsilon_n$,). So it is a set of orthogonal bases.

Sufficiency: push the necessity backwards.

3.2 The quadratic form is transformed into a standard form by using an orthogonal matrix

In many cases, one often needs to use the orthogonal linear substitution of the general quadratic form as the standard form, which links the orthogonal matrix with the quadratic form.

The following is a general step [5] of transforming the orthogonally transformed quadratic form into a standard form:

First, the corresponding real $f(x_1,x_2,...,x_s)$ symmetric matrix is obtained $A=(a_{ij})_{s\times s}$ from the known quadratic type;

Secondly, according A to the corresponding characteristic polynomial, the corresponding characteristic vector of each characteristic value is solved, orthogonalization, and unity;

The various feature vectors are arranged Q to form an orthogonal matrix, satisfies

$$Q^TAQ=Q^{-1}AQ=\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_s\end{pmatrix}$$

Finally, remember the $X=QY$ orthogonal $f(x_1,x_2,...,x_s)=\lambda_1y_1^2+\lambda_2y_2^2+\cdots+\lambda_sy_s^2$ transformation to make the.

3.2.1 Specific methods and application examples of the symmetry matrix diagonalization problem

Here we mainly discuss the real symmetric matrix, combined with the correlation properties of the real symmetric matrix, turn the real symmetric matrix to diagonal type, and the main methods involved are Schmidt orthogonalization and unualization.

Remember the $A=(a_{ij})_{n\times n}$ symmetric matrix, the following step [5] is orthogonally similar to the diagonal matrix:

First, the different A eigenvalues and the corresponding linear independent eigenvectors $\lambda_1,\lambda_2,...,\lambda_s$ should A be set as all the different eigenvalues, $\gamma_1,\gamma_2,...,\gamma_s$ and $\gamma_1+\gamma_2+\cdots+\gamma_s=n$ then set their λ_i weights γ_i as respectively. And $\alpha_{i1}, \alpha_{i1}, \ldots, \alpha_{ir_i}, (i = 1, 2, \ldots s)$ set the individual eigenvectors corresponding to the eigenvalue as respectively,

Secondly, the $\gamma_i > 1$ feature vectors $\alpha_{i1}, \alpha_{i1}, \ldots, \alpha_{ir_i}$ are Schmidt orthogonalized:

$$, \beta_{i1} = \alpha_{i1} \quad , \beta_{ij} = \alpha i_j - \frac{(\alpha_{ij}, \beta_{i1})}{(\beta_{i1}, \beta_{i1})} \beta_{i1} - \ldots - \frac{(\alpha_{ij}, \beta_{ij-1})}{(\beta_{ij-1}, \beta_{ij-1})} \beta_{ij-1} \ (j = 1, 2, \ldots, \gamma_i)$$

Then the unalization process:, $\chi_{ij} = \frac{1}{\|\beta_{ij}\|} \beta_{ij}$ $\chi_{ij} = \frac{1}{\|\beta_{ij}\|} \beta_{ij}$

If, directly $\gamma_i = 1$ unalize β_{i1}, i. e. $\chi_{i1} = \frac{1}{\|\beta_{i1}\|} \beta_{i1}$

Finally, the orthogonal matrix is constructed based on the resulting orthogonal unit feature vectors

$$. X = (\chi_{11}, \chi_{12}, \ldots, \chi_{1r_1}, \chi_{21}, \chi_{22}, \ldots, \chi_{2r_2}, \ldots, \chi_{s1}, \chi_{s2}, \ldots, \chi_{sr_s})$$

Is

$$. Q^{-1} A Q = Q^T A Q = \begin{pmatrix} \lambda_1 E_{r_1} & O & \cdots & O \\ O & \lambda_2 E_{r_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \lambda_s E_{r_s} \end{pmatrix}$$

Practical examples below show how to turn a symmetric matrix into a diagonal type with an orthogonal matrix.

Example 5 has a second $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ order $B = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$ matrix, and try the orthogonal matrices into diagonal types P, and find $P^T A P = B$ the orthogonal matrix, making.

Solution: Starting from the characteristic $|\lambda E - A| = \begin{vmatrix} \lambda - 1 & 2 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda - 3)(\lambda + 1)$ polynomial A, yes $\lambda_1 = 3$, $\lambda_2 = -1$ then the eigenvalue is.

Solve the system of homogeneous $(\lambda_i E - A) x = 0, i = 1, 2$ linear equations in turn:

It is concluded that the unit feature $p_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ vector $p_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ is $P_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$:,,

remember $P_1^T A P_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$, so, there is.

Likewise, $|\mu E - B| = \begin{vmatrix} \lambda - 2 & -\sqrt{3} \\ -\sqrt{3} & \lambda \end{vmatrix} = (\mu - 3)(\mu + 1)$ the solved B eigenvalue $\mu_1 = 3$ are $\mu_2 = -1$,

Similar A steps, get,, remember $p_3 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$, $p_4 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ then $P_2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. $P_2^T B P_2 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

To sum up, $P_1^T A P_1 = P_2^T B P_2$ yes, to the left type of $P_2 P_1^T A P_1 P_2^T = B$ deformation $P = P_1 P_2^T$, get, so,

$P^T AP = B$ remember, and thus get.

Example 6 has a 3rd order $A = \begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$ matrix, and a $\beta = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ 3 D column vector $Ax = \beta$. If the linear system a of equations Q has infinite $Q^T AQ$ many solutions, the value and orthogonal matrix make it diagonal matrix.

Solution: transform the enlargement A matrix of the coefficient (A, β) matrix of the linear system

$$(A, \beta) = \begin{pmatrix} 1 & 1 & a & 1 \\ 1 & a & 1 & 1 \\ a & 1 & 1 & -2 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & (a-1)(a+2) & a+2 \end{pmatrix}$$

Combined with the problem known, the linear equation $r(A) = r(A, \beta) < 3$ has infinite $a = -2$ many solutions, so that.

Thus, the characteristic $A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$ polynomial A obtained is:

$$, |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & 2 \\ -1 & \lambda + 2 & -1 \\ 2 & -1 & \lambda - 1 \end{vmatrix} = \lambda (\lambda - 3)(\lambda + 3)$$

The A eigenvalue $\lambda_1 = 3$ is $\lambda_2 = -3$,, $\lambda_3 = 0$,, subsequent solution of homogeneous linear equations:

$$(\lambda_i E - A)x = 0, i = 1, 2, 3$$

Unitizes the resulting feature vector, get

$$, q_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad q_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}, q_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Remember $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$, so, I $Q^T AQ = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have:.

For a system of equations $Ax = b$, if the coefficient matrix A is used as an orthogonal matrix, then the above system has a unique solution.

3.2.2 Application examples of quadratic surface equation under orthogonal matrix simple rectangular coordinate system

The problem of simplifying the quadratic surface equation is also a kind of question that replaces the quadratic type as the standard form. It requires scholars to turn the surface equation into the general form of quadratic type, and then reasonably use the shifting axis and rotating axis to transform the surface equation into the standard equation of quadratic type. In the process of investigation, it still reflects the proficiency of scholars in the use of orthogonal matrix properties. The following is described from specific examples.

Example 7: simplify the quadratic $6x^2+5y^2+7z^2-4xy+4xz+12x+6y+18z=0$ surface equation, judge which kind of surface it belongs to, and write the corresponding rectangular coordinate transformation formula.

Analysis: For similar problems, the cross product can be eliminated by a certain rotation, and the corresponding translation into the standard equation, so as to achieve the purpose of simplification. Among them, the rotation of this process is equivalent to an orthogonal transformation, therefore, the problem only consider the first part of the left side of the equation, the quadratic term as a quadratic type, through orthogonal linear substitution quadratic type to standard shape, then simplify the quadratic surface equation by translation into a standard equation, can solve the problem [5].

Solution: Starting from the real symmetry matrix A corresponding to the left part of the equation, starting from the meaning of the question

$$A=\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}$$

Its characteristic polynomial $|\lambda E-A|=\begin{vmatrix} \lambda-6 & 2 & -2 \\ 2 & \lambda-5 & 0 \\ -2 & 0 & \lambda-7 \end{vmatrix}=(\lambda-3)(\lambda-6)(\lambda-9)$ is A :, so the eigenvalue $\lambda_1=3$ solution $\lambda_2=6$:, $\lambda_3=9$,.

Solve homogeneous linear $(\lambda_i E-A)x=0, i=1,2,3$ equations in turn, and the corresponding eigenvector corresponding to each eigenvalue are:

$$,p_1=\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \quad ,p_2=\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad p_2=\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

To sum up, the orthogonal $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ transformation:, after, take the orthogonal transformation $3x'^2+6y'^2+9z'^2+6x'+12y'+18z'=0$ into the surface, simplify, get, formula, get:

$$.3(x'+1)^2+6(y'+1)^2+9(z'+1)^2=18$$

Order $\begin{cases} x'=x''-1 \\ y'=y''-1 \\ z'=z''-1 \end{cases}$, the resulting surface standard $\frac{x''^2}{6}+\frac{y''^2}{3}+\frac{z''^2}{2}=1$ equation is:, this surface is ellipsoid, the rectangular coordinate transformation formula used in the above simplification process is:

$$\begin{cases} x=\frac{2}{3}x''-\frac{1}{3}y''+\frac{2}{3}z''-1 \\ y=\frac{2}{3}x''+\frac{2}{3}y''-\frac{1}{3}z''-1 \\ z=-\frac{1}{3}x''+\frac{2}{3}y''+\frac{2}{3}z''-1 \end{cases}$$

As shown in the above example, before moving the axis and turning the axis, the general equation of the secondary surface is transformed into a familiar quadratic form, so as to then replace the quadratic form as the standard form. Finally, the transformation formula is brought into the original equation to achieve the purpose of simplification. The method used for the quadratic surface equation simplification, the same example 11 is completely consistent, so the other similar questions will not be mentioned here.

3.3 Partial application of orthogonal matrix in matrix decomposition

There are many forms of decomposition for the reversible matrix, and it can be generalized from the decomposition forms of the reversible matrix to the general real matrix. The following will $Q-R$ start with three decomposition forms related to the orthogonal matrix. Polar decomposition, decomposition, and singular value decomposition, respectively.

(1) Extreme decomposition

For any first-order n reversible A real matrix, we prove that there S is a positive definite Q matrix $A=QS$ and an orthogonal matrix, so that this decomposition is unique. This decomposition is called the pole decomposition of [6].

Proof: First, prove the existence.

As reversible A , and $A^T A$ hence a positive definite matrix. According to the properties S of the positive definite matrix, there is a positive definite matrix, such that:

$$\cdot A^T A = S^2$$

thus. remember $A=\left(A^T\right)^{-1}S^2=\left(A^T\right)^{-1}SS$. $Q=\left(A^T\right)^{-1}S$ owing to:

$$QQ^T=\left(A^T\right)^{-1}SSA^{-1}=\left(A^T\right)^{-1}A^TAA^{-1}=E$$

So the matrix Q is an orthogonal matrix.

And then we can prove the uniqueness. set up

, Where, is the orthogonal $A=Q_1S_1=Q_2S_2$ matrix Q_1, Q_2 and is the S_1 positive S_2 definite matrix. thereupon

$$,\left(Q_1S_1\right)^T Q_1S_1=S_1^2 \quad ,\left(Q_2S_2\right)^T Q_2S_2=S_2^2$$

thus

$$S_1^2=S_2^2=A^TA$$

According A^TA, S_1, S_2 to the positive definite matrix, then combined with the corresponding properties of the positive definite matrix, there are

$$,S_1=S_2$$

thus

$$\cdot Q_1=AS_1^{-1}=AS_2^{-1}=Q_2$$

To sum up, the proposition is a proof.

(2) decompose $Q-R$

Example 9 For any reversible A matrix, we can start from the matrix decomposition and uniquely Q decompose it as the product 0 of an orthogonal R matrix and an upper $Q-R$ triangular matrix where the main diagonal elements are larger than. This decomposition is called a decomposition.

Evidence: starting A from the general form $A=\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$ of the $\alpha_i\left(i=1,2,\ldots,n\right)$ matrix n, remember, where, is the dimensional column vector.

The column vector A is linearly irrelevant as known from the title, with Schmidt orthogonalization below:

$$,\beta_1=\alpha_1$$

$$,\beta_2=\alpha_2-\frac{\left(\alpha_2,\beta_1\right)}{\left(\beta_1,\beta_1\right)}\beta_1$$

$$\vdots$$

$$,\beta_i=\alpha_i-\frac{\left(\alpha_i,\beta_1\right)}{\left(\beta_1,\beta_1\right)}\beta_1-\frac{\left(\alpha_i,\beta_2\right)}{\left(\beta_2,\beta_2\right)}\beta_2-\cdots-\frac{\left(\alpha_i,\beta_{i-1}\right)}{\left(\beta_{i-1},\beta_{i-1}\right)}\beta_{i-1}\ i=2,3,\ldots,n$$

By resort to deformation, say:

$$,\alpha_1=\beta_1$$

$$\alpha_2=\beta_2+\frac{\left(\alpha_2,\beta_1\right)}{\left(\beta_1,\beta_1\right)}\beta_1$$

$$\vdots$$

$$\alpha_i=\frac{\left(\alpha_i,\beta_1\right)}{\left(\beta_1,\beta_1\right)}\beta_1+\frac{\left(\alpha_i,\beta_2\right)}{\left(\beta_2,\beta_2\right)}\beta_2+\ldots+\frac{\left(\alpha_i,\beta_{i-1}\right)}{\left(\beta_{i-1},\beta_{i-1}\right)}\beta_{i-1}+\beta_i$$

get

$$,A=\left(\beta_1,\beta_2,\ldots,\beta_n\right)\begin{pmatrix}1 & & & * \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1\end{pmatrix}$$

After the orthogonal vector $\beta_1,\beta_2,\ldots,\beta_n$ groups unalized, get

$$,q_i=\frac{1}{\left\|\beta_i\right\|}\beta_i\quad .i=1,2,\ldots,n$$

So, there is $A=\left(q_1,q_2,\ldots,q_n\right)\begin{pmatrix}\left\|\beta_1\right\| & & & * \\ 0 & \left\|\beta_2\right\| & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \left\|\beta_n\right\|\end{pmatrix}$,

Note $Q=\left(q_1,q_2,\ldots,q_n\right)$,, $R=\begin{pmatrix}\left\|\beta_1\right\| & & & * \\ 0 & \left\|\beta_2\right\| & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \left\|\beta_n\right\|\end{pmatrix}$ thus the orthogonal Q matrix, and the upper triangular matrix R whose $A=QR$ main diagonal element is positive, so.

The following Q,R proof of uniqueness:

Assuming the existence of Q' another orthogonal R' matrix $A=Q'R'$, as well as the upper triangle matrix, makes, then, have

, $A=Q'R'=QR$ The left matrix $Q^{-1}Q'=RR'^{-1}$ is both the orthogonal matrix, and the upper triangular matrix, according to the properties of the orthogonal matrix, this matrix ±1 is a diagonal R' matrix R, so the main diagonal

$Q^{-1}Q' = RR'^{-1}$ element $Q = Q'$ of $R = R'$ the matrix is, and because the diagonal elements are both positive, so, yes, that is, thus unique.

On the basis of this decomposition, there can be a more general $A_{m\times n}$ form, that is, $m > n$ () when the matrix is a column full-rank matrix $Q_{m\times n}$, there is a matrix where the column vector $R_{n\times n}$ group $A_{m\times n} = Q_{m\times n}R_{n\times n}$ is orthogonal unit vector group and the upper triangular matrix where the main diagonal elements are all positive, making that. The proof of this conclusion is similar to the above proof process, and it will not be repeated here. Further issues can also be resolved from this conclusion, as follows:

Example 10 assumes a matrix of full $A_{m\times n}$ rank (subsequent A short notation) that can be decomposed into a matrix $m\times n$ where Q a group of column vectors are orthogonal unit R vectors and $A = QR$ a product of a superior triangular R^m matrix α where the main $A^T AX = A^T\alpha$ diagonal $R^{-1}Q^T\alpha$ elements are positive, i. e. According to this condition, it is proved that the unique solution of the system of linear equations is.

Evidence: Consider the column Q vector group $\eta_1,\eta_2,\ldots,\eta_n$ of the matrix

$$,Q^T Q = \begin{pmatrix} \eta_1^T \\ \eta_2^T \\ \vdots \\ \eta_n^T \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \end{pmatrix} = \begin{pmatrix} \eta_1^T\eta_1 & \eta_1^T\eta_2 & \cdots & \eta_1^T\eta_n \\ \eta_2^T\eta_1 & \eta_2^T\eta_2 & \cdots & \eta_2^T\eta_n \\ \vdots & \vdots & & \vdots \\ \eta_n^T\eta_1 & \eta_n^T\eta_2 & \cdots & \eta_n^T\eta_n \end{pmatrix} = E$$

thereupon

$$A^T A\left(R^{-1}Q^T\alpha\right) = (QR)^T (QR)\left(R^{-1}Q^T\alpha\right) = R^T Q^T QRR^{-1}Q^T\alpha$$
$$= R^T Q^T\alpha = (QR)^T \alpha = A^T\alpha$$

To sum up, there $A^T AX = A^T\alpha$ is: the only $R^{-1}Q^T\alpha$ solution is.

Example 11 $Q-R$ solves the system of $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 2 & -5 & 8 \\ -1 & 3 & -7 \end{pmatrix} x = \begin{pmatrix} -1 \\ 1 \\ 11 \\ 9 \end{pmatrix}$ linear equations by decomposition

Solution: remember

$$,A = (\alpha_1,\alpha_2,\alpha_3) = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 2 & -5 & 8 \\ -1 & 3 & -7 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 1 \\ -11 \\ 9 \end{pmatrix}$$

Schmidt A orthogonization $\alpha_1,\alpha_2,\alpha_3$ of the three column vectors of the matrix, and get

$$\begin{cases} \beta_1 = \alpha_1 = (1,2,2-1)^T, \\ \beta_2 = \alpha_2 - \dfrac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1 = (2,3,-3,2)^T, \\ \beta_3 = \alpha_3 - \dfrac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1 - \dfrac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2 = (1,2,2,-1)^T, \end{cases}$$

Then unit each component, get

$$\eta_1=\frac{1}{\sqrt{10}}\begin{pmatrix}1\\2\\2\\-1\end{pmatrix},\eta_2=\frac{1}{\sqrt{20}}\begin{pmatrix}2\\3\\-3\\2\end{pmatrix},\eta_3=\frac{1}{\sqrt{10}}\begin{pmatrix}2\\-1\\-1\\-2\end{pmatrix},$$

Remember, by the orthogonal unit column $Q=(\eta_1,\eta_2,\eta_3)$ vector Q group, get, and $Q^TQ=E$ then $Q-R$ according $A=QR$ to the decomposition, get

So, there is

$$.R=Q^TQR=Q^TA$$

Bring in the corresponding data and simplify it

$$=\begin{pmatrix}\frac{1}{\sqrt{10}}&\frac{2}{\sqrt{20}}&\frac{2}{\sqrt{10}}&-\frac{1}{\sqrt{10}}\\\frac{2}{\sqrt{26}}&\frac{3}{\sqrt{26}}&-\frac{3}{\sqrt{26}}&\frac{2}{\sqrt{26}}\\\frac{2}{\sqrt{10}}&-\frac{1}{\sqrt{10}}&-\frac{1}{\sqrt{10}}&-\frac{2}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}1&1&3\\2&1&2\\2&-5&8\\-1&3&-7\end{pmatrix},$$

$$=\begin{pmatrix}\sqrt{10}&-\sqrt{10}&3\sqrt{10}\\&\sqrt{26}&-\sqrt{26}\\&&\sqrt{10}\end{pmatrix}$$

Further, yes

$$R^{-1}=\begin{pmatrix}\frac{1}{\sqrt{10}}&\frac{1}{\sqrt{26}}&-\frac{2}{\sqrt{10}}\\&\frac{1}{\sqrt{26}}&\frac{1}{\sqrt{10}}\\&&\frac{1}{\sqrt{10}}\end{pmatrix}$$

Integrating the solutions $x=R^{-1}Q^Tb$ of the available system of equations

$$=\begin{pmatrix}\frac{1}{\sqrt{10}}&\frac{1}{\sqrt{26}}&-\frac{2}{\sqrt{10}}\\&\frac{1}{\sqrt{26}}&\frac{1}{\sqrt{10}}\\&&\frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{10}}&\frac{2}{\sqrt{20}}&\frac{2}{\sqrt{10}}&-\frac{1}{\sqrt{10}}\\\frac{2}{\sqrt{26}}&\frac{3}{\sqrt{26}}&-\frac{3}{\sqrt{26}}&\frac{2}{\sqrt{26}}\\\frac{2}{\sqrt{10}}&-\frac{1}{\sqrt{10}}&-\frac{1}{\sqrt{10}}&-\frac{2}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}-1\\1\\-11\\9\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt{10}}&\frac{1}{\sqrt{26}}&-\frac{2}{\sqrt{10}}\\&\frac{1}{\sqrt{26}}&\frac{1}{\sqrt{10}}\\&&\frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}-2\sqrt{10}\\2\sqrt{26}\\-\sqrt{10}\end{pmatrix}=\begin{pmatrix}1\\1\\-1\end{pmatrix}.$$

Finally, it $x=\begin{pmatrix}1\\1\\-1\end{pmatrix}$ is brought into the original system of equations, and the equation is the solution to the system of equations.

(3) singular value decomposition

First give the concept of singular value [6]: r with $m\times n$ a real A matrix $A^T A$ of rank of, the arithmetic A root of the positive eigenvalue is called the singular value of the matrix.

Example 12 Set any real reversible A matrix, and prove U,P that there is an orthogonal matrix, making the

$$,UAP=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}$$

Where $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the A singular value of the matrix.

Evidence: According to the above pole decomposition process R, an orthogonal matrix S and $A=RS$ a positive definite matrix can be obtained.

Combined S with a positive definite matrix, B so, there is an orthogonal matrix, so that the following formula holds:

That $B^T SB=\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}$ is, all $\lambda_1,\lambda_2,\ldots,\lambda_n$ are greater 0 than, so

$$,B^{-1}R^{-1}AB=B^{-1}SB=\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}$$

Order $B^{-1}R^{-1}=U$, remember $P=B$, yes

$$,UAP=\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}$$

again

$$A^T A=\left(P^{-1}\right)^T\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}\left(U^{-1}\right)^T U^{-1}\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}P^{-1}$$

approach

$$,A^T A=P\begin{pmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}^2 P^{-1}$$

Therefore $\lambda_1, \lambda_2, \ldots, \lambda_n$ a singular A value of the matrix, thus the proposition.

Further extension of this case can be obtained: for $m \times n$ any real A matrix m, there are Q order n orthogonal U matrix and order orthogonal matrix, so that

$$, \text{among } QAU = \begin{pmatrix} D & O \\ O & O \end{pmatrix} . D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix}$$

Analysis: For the known real A matrix, it is first transformed into an equivalent standard form, and the two reversible matrices $Q - R$ in the elementary transformation process are moved to the right end of the equation. Combined with the decomposition, the reversible matrix is decomposed first, and then brought back to the original formula. Combined with the proof process of example 16, this proposition can be proved. It will not be repeated here.

4 Conclusion

Orthogonal matrix has many applications in the higher algebraic matrix and Euclidean space parts, and plays a great role in solving some difficult problems with positive definite matrix. In addition to the application of higher algebra, orthogonal matrix also plays a certain role in recent algebra, physics and chemistry, so it is necessary to learn orthogonal matrix well. The key to learning orthogonal matrix lies in the skilled use of orthogonal matrix correlation properties and the specific use of some aspects. This article introduces nine common properties of orthogonal matrix, and three applications, which can let scholars better use orthogonal matrix to solve problems.

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